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EXTENSIONS OF ORDERED SETS HAVING THE FINITE CUTSET PROPERTY

John GINSBURG*
Department of Mathematics, University of Winnipeg, Winnipeg, Manitoba, Canada R3B 2E9

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Let P be an ordered set. P is said to have the finite cutset property if for every x in P there is a finite set F of elements which are noncomparable to x such that every maximal chain in P meets $\{x\} \cup F$. It is well known that this property is equivalent to the space of maximal chains of P being compact. We consider the following question: Which ordered sets P can be embedded in an ordered set Q which has the finite cutset property?

For $x \in P$, we let $x^+ = \{p \in P: x \leqslant p\}$. Our main results are the following:

Theorem. *Suppose P can be embedded in an ordered set having the finite cutset property and that A is an uncountable antichain in P. Then there exist distinct elements a, b, c in A such that $a^+ \cap b^+ = a^+ \cap c^+$.*

Corollary. *There exists an ordered set P which cannot be embedded in an ordered set having the finite cutset property whereas every countable subset of P can be embedded in such an ordered set.*

1. A necessary condition for embeddability in an ordered set having the finite cutset property

Let P be an ordered set. We let $\mathcal{M}(P)$ denote the set of maximal chains in P. We endow $\mathcal{M}(P)$ with the induced topology it inherits as a subspace of 2^P, and refer to $\mathcal{M}(P)$ as the space of maximal chains of P. Conditions on P which are equivalent to the space $\mathcal{M}(P)$ being compact are derived in [1]. Those conditions involve the following concepts. For $x \in P$ we let $I(x) = \{p \in P: p$ is noncomparable to $x\}$. A subset S of P is called a *cutset* for P if every maximal chain in P intersects S.

Theorem 1.1 ([1]). *Let P be an ordered set. Then the following are equivalent:*
(i) *$\mathcal{M}(P)$ is compact.*
(ii) *For every $x \in P$ there is a finite set $F \subseteq I(x)$ such that $\{x\} \cup F$ is a cutset for P.*

Following [5], if P satisfies condition (ii) in Theorem 1.1 we say that P has the *finite cutset property,* and if F is a subset of $I(x)$ for which $\{x\} \cup F$ is a cutset we will say that $\{x\} \cup F$ is a *cutset for x* in P.

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There are two natural kinds of questions which may be asked concerning the finite cutset property and the compactness of $\mathcal{M}(P)$. Firstly, one may inquire as to which compact spaces can be represented in the form $\mathcal{M}(P)$ for some ordered set P. Results in that direction are obtained in [1] and [5]. Secondly one may ask how extensive is the class of ordered sets having the finite cutset property. It is clear that this is a very restricted class of ordered sets. In [5] and [6] the size of antichains in such ordered sets is studied, and in [3] natural conditions on P are derived which imply that P has the finite cutset property. One way to determine how extensive is the class of ordered sets having the finite cutset property is by describing exactly which ordered sets P can be embedded in an ordered set with the finite cutset property. Ideally we would like to have a minimal list of 'forbidden configurations'—a minimal list of ordered sets which cannot be embedded in an ordered set having the finite cutset property—having the property that any ordered set which cannot be embedded in an ordered set with the finite cutset property must contain a copy of one of the ordered sets in the list. Although we have been unable to obtain such a minimal list, we will obtain some information in that direction.

The finite cutset property is completely determined by countable subsets, as indicated by the following observation from [3].

Theorem 1.2 [3]. *Let P be an ordered set.*

(i) *If P has the finite cutset property and S is any countable subset of P, then there is a countable subset P_1 of P such that $S \subseteq P_1$ and P_1 has the finite cutset property.*

(ii) *If P does not have the finite cutset property, then there is a countable subset P_1 of P and an element $x \in P_1$ such that x does not have a finite cutset in P_1.*

In the light of Theorem 1.2 one might expect that whether or not a given ordered set P could be embedded in an ordered set, having the finite cutset property, is determined completely by the countable subsets of P. In fact this turns out to be false, and in Section 2 we exhibit examples of ordered sets P which cannot be embedded in an ordered set with the finite cutset property but for which every countable subset can be so embedded.

Before proceeding with these matters we would like to point out two examples which illustrate the above ideas and to which we will later refer. For any cardinal number κ, let P be an antichain of cardinality κ. Say $P = \{a_\alpha : \alpha < \kappa\}$. Although P does not have the finite cutset property, it is a simple matter to extend P to a larger ordered set which does have this property. Such an extension appears in Fig. 1. Note that in this figure every element has a cutset consisting of at most 3 elements.

As a second example consider $P = \{x_n : n \in \omega\} \cup \{y_n : n \in \omega\}$, where $\{x_n : n \in \omega\}$ and $\{y_n : n \in \omega\}$ are antichains and where $x_n < y_m \leftrightarrow n \neq m$. Figure 2 shows an extension of P having the finite cutset property.

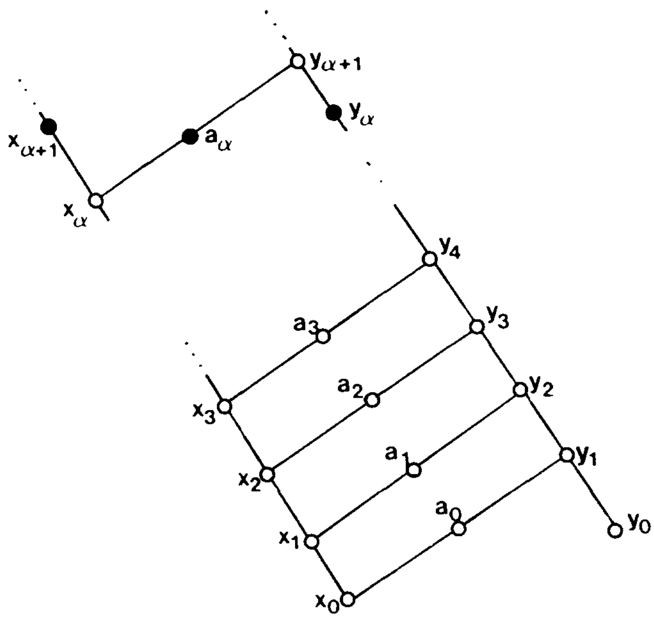

Fig. 1. This figure shows an ordered set having the finite cutset property in which the antichain $\{a_\alpha : \alpha < \kappa\}$ is embedded. The three shaded elements $\{x_{\alpha+1}, a_\alpha, y_\alpha\}$ form a three element cutset for a_α.

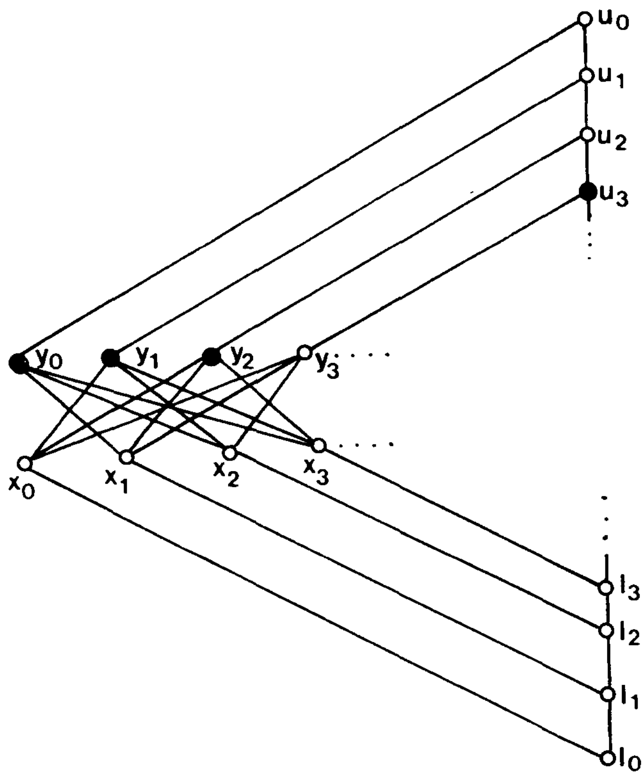

Fig. 2. The figure shows an ordered set having the finite cutset property in which $P = \{x_n : n \in \omega\} \cup \{y_n : n \in \omega\}$ is embedded. The shaded elements $\{y_0, y_1, y_2, u_3\}$ form a finite cutset for y_2.

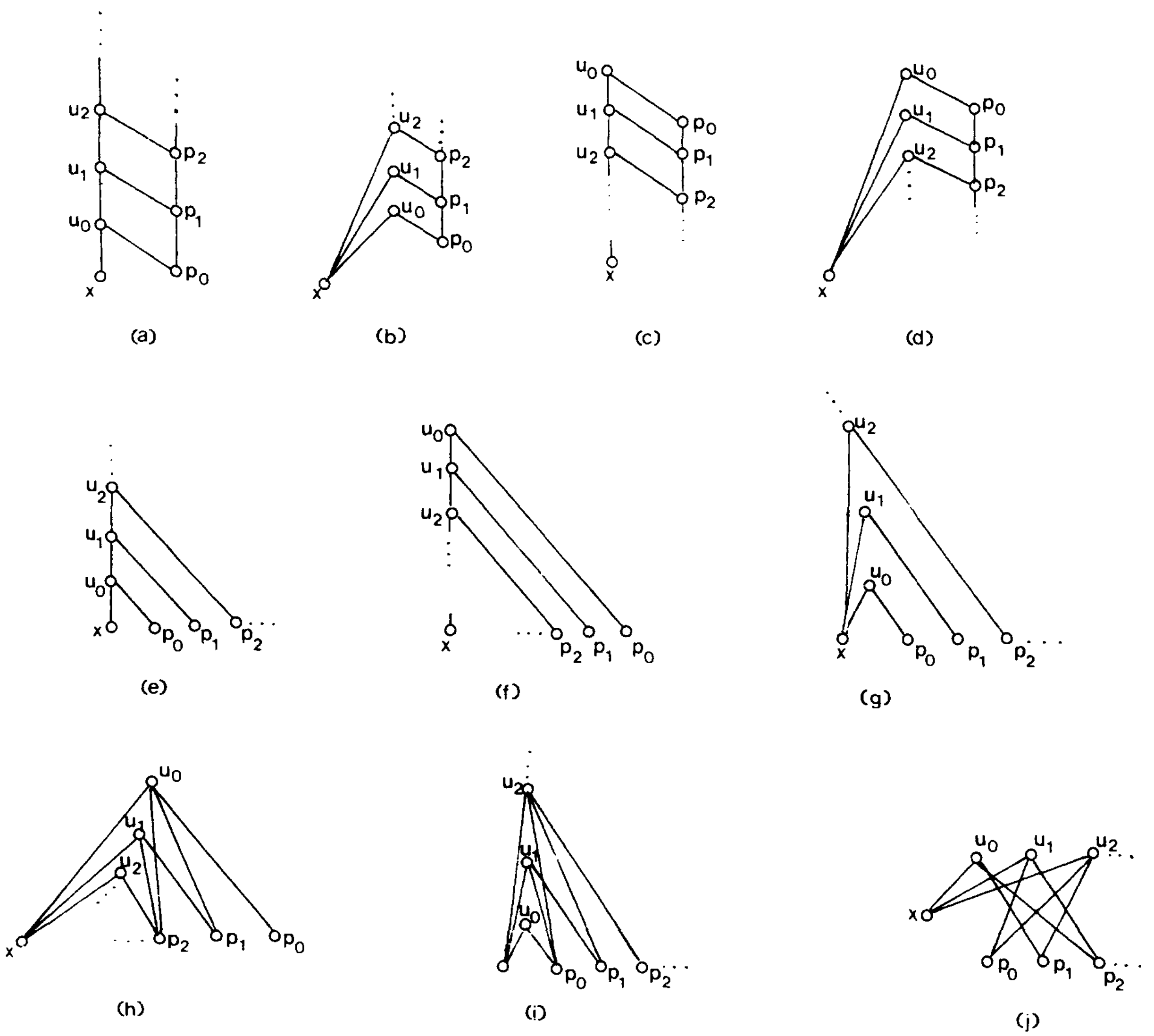

Fig. 3. Each of the ordered sets drawn here is of the form $P=\{x\}\cup\{p_n\colon n\in\omega\}\cup\{u_n\colon n\in\omega\}$, where each p_n is noncomparable to x, and where $x<u_n$ for each n. In the last three of these, $\{p_n\colon n\in\omega\}$ and $\{u_n\colon n\in\omega\}$ are antichains, and we have respectively, from left to right, $p_n<u_m$ iff $m\leqslant n$, $n\leqslant m$, $m\neq n$.

We next set out to obtain a necessary condition for the embeddability of an ordered set P in an ordered set having the finite cutset property. We will first describe some ordered sets which cannot be so embedded.

Specifically, we show in the following lemma that if P has the form $P=\{x\}\cup\{p_n\colon n\in\omega\}\cup\{l_n\colon n\in\omega\}\cup\{u_n\colon n\in\omega\}$, where $\{x\}\cup\{p_n\colon n\in\omega\}\cup\{u_n\colon n\in\omega\}$ is isomorphic to one of the ordered sets in Fig. 3 and where $\{x\}\cup\{p_n\colon n\in\omega\}\cup\{l_n\colon n\in\omega\}$ is isomorphic to the dual of one of the ordered sets in Fig. 3, then P cannot be embedded in any ordered set which has the finite cutset property. Figure 4 shows some of these ordered sets P.

Lemma 1.3. *Let P be an ordered set of the form*

$$P=\{x\}\cup\{p_n\colon n\in\omega\}\cup\{l_n\colon n\in\omega\}\cup\{u_n\colon n\in\omega\},$$

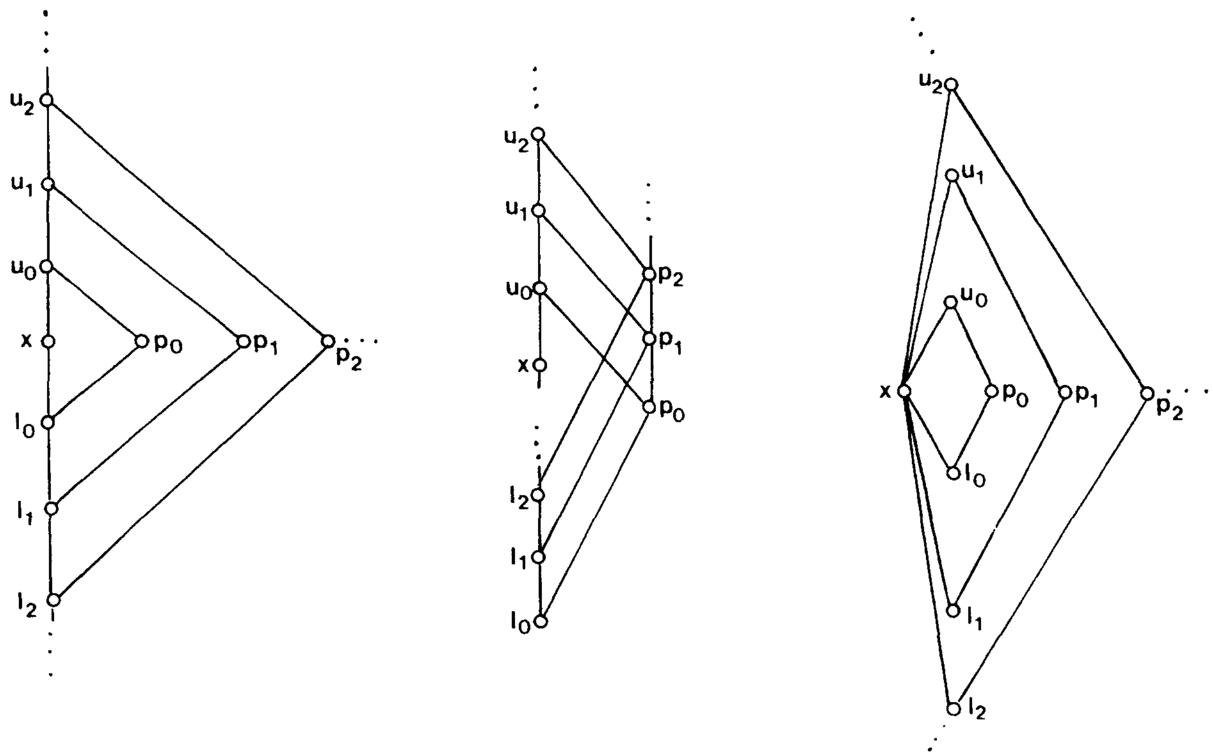

Fig. 4. Here we illustrate three of the ordered sets of the form $P = \{x\} \cup \{p_n : n \in \omega\} \cup \{l_n : n \in \omega\} \cup \{u_n : n \in \omega\}$, where $\{x\} \cup \{p_n : n \in \omega\} \cup \{u_n : n \in \omega\}$ is isomorphic to one of the ordered sets in Fig. 3 and $\{x\} \cup \{p_n : n \in \omega\} \cup \{l_n : n \in \omega\}$ is isomorphic to the dual of one of the ordered sets in Fig. 3.

where $\{x\} \cup \{p_n : n \in \omega\} \cup \{u_n : n \in \omega\}$ is isomorphic to one of the ordered sets in Fig. 3 and $\{x\} \cup \{p_n : n \in \omega\} \cup \{l_n : n \in \omega\}$ is isomorphic to the dual of one of the ordered sets in Fig. 3. Then P cannot be embedded in an ordered set having the finite cutset property.

Proof. Let Q be any ordered set containing P. We will show that Q does not have the finite cutset property by showing that x does not have a finite cutset in Q. We argue by contradiction. Thus suppose x does have a finite cutset in Q. Then there is a finite set F of elements of Q which are noncomparable to x and such that every maximal chain in Q meets $\{x\} \cup F$. Now, we let $P^+ = \{x\} \cup \{p_n : n \in \omega\} \cup \{u_n : n \in \omega\}$, and we let $P^- = \{x\} \cup \{p_n : n \in \omega\} \cup \{l_n : n \in \omega\}$. We now distinguish three cases, depending on whether P^+ is isomorphic to Fig. 3(j) or P^- is isomorphic to the dual of Fig. 3(j) or both.

Case 1. Suppose P^+ is not isomorphic to Fig. 3(j) and P^- is not isomorphic to the dual of Fig. 3(j). Then, for each n, let M_n be a maximal chain in Q such that $\{u_n, p_n, l_n\} \subseteq M_n$. Since p_n is noncomparable to x we have that $x \notin M_n$. But $\{x\} \cup F$ is a cutset for Q, and so $M_n \cap F \neq \emptyset$. For each $f \in F$, we let $S(f) = \{n \in \omega : f \in M_n\}$. Since F is finite, one of the sets $S(f)$ must be infinite. So suppose that f is an element of F for which $S(f)$ is infinite. If $n \in S(f)$, then $f \in M_n$ and so $\{f, u_n, p_n, l_n\}$ is a chain. But f is noncomparable to x and so we cannot have either

$f \leqslant l_n$ or $f \geqslant u_n$. Therefore we either have that $l_n \leqslant f \leqslant p_n$ or $p_n \leqslant f \leqslant u_n$. We let $S_1 = \{n \in S(f): l_n \leqslant f \leqslant p_n\}$ and we let $S_2 = \{n \in S(f): p_n \leqslant f \leqslant u_n\}$. One of the sets S_1, S_2 is infinite. We assume that S_2 is infinite; we argue dually if S_1 is infinite. Let n, m be any two elements of S_2 with $n < m$. Then we have $p_n \leqslant f \leqslant u_n$ and $p_m \leqslant f \leqslant u_m$. Hence $p_n \leqslant f \leqslant u_m$ and $p_m \leqslant f \leqslant u_n$. But in any of the figures in Fig. 3 other than Fig. 3(j), if $n < m$, then either $p_n \nleqslant u_m$ or $p_m \nleqslant u_n$. This is a contradiction.

Case 2. Suppose that either P^+ is isomorphic to Fig. 3(j) or that P^- is isomorphic to the dual of Fig. 3(j), but not both. There are two dual cases involved here. We will give the argument for the case in which P^+ is isomorphic to Fig. 3(j) and P^- is not isomorphic to the dual of Fig. 3(j). Thus we have $p_n \leqslant u_m \leftrightarrow n \neq m$. In this case we will apply Ramsey's theorem (see [4]). If $n, m \in \omega$ and $n < m$, we choose a maximal chain $M_{n,m}$ in Q such that $\{l_n, p_n, u_m\} \subseteq M_{n,m}$. Now x cannot be in $M_{n,m}$ because x is noncomparable to p_n and so $M_{n,m} \cap F \neq \emptyset$. We partition the pairs (n, m) of elements of ω with $n < m$ into boxes B_f, for $f \in F$, by letting $B_f = \{(n, m): n < m$ and $f \in M_{n,m}\}$. By Ramsey's theorem there is an infinite subset S of ω and an element $g \in F$ such that for any two elements n, m of S with $n < m$ we have $g \in M_{n,m}$. Now if $g \in M_{n,m}$, then $\{g, l_n, p_n, u_m\}$ is a chain. But g is noncomparable to x and so we can't have either $g \leqslant l_n$ or $u_m \leqslant g$. Therefore either $l_n \leqslant g \leqslant p_n$ or $p_n \leqslant g \leqslant u_m$. Now, as the proof of Case 1 shows, there is at most one n for which $l_n \leqslant g \leqslant p_n$ (because if $n < m$, then either $l_n \nleqslant p_m$ or $l_m \nleqslant p_n$). Discarding this n from S together with all members of S, which are $\leqslant n$, we obtain an infinite subset S_1 of S having the property that, for all $n, m \in S_1$ with $n < m$; $p_n \leqslant g \leqslant u_m$. Let n, m, k be any three elements of S_1 with $n < m < k$. Then $p_n \leqslant g \leqslant u_m$ and $p_m \leqslant g \leqslant u_k$. Hence $p_m \leqslant g \leqslant u_m$. But this contradicts the fact that $p_m \nleqslant u_m$ in Fig. 3(j).

Case 3. Suppose that P^+ is isomorphic to Fig. 3(j) and that P^- is isomorphic to the dual of Fig. 3(j). In this case we have $l_n \leqslant p_m \leftrightarrow n \neq m$ and $p_n \leqslant u_m \leftrightarrow n \neq m$. We argue similarly to Case 2, using triples rather than pairs. If $n < m < k$ we choose a maximal chain $M_{n,m,k}$ in Q containing the chain $\{l_n, p_m, u_k\}$. We partition the set of all triples (n, m, k) with $n < m < k$ into boxes B_f, for $f \in F$, by letting $B_f = \{(n, m, k): n < m < k$ and $f \in M_{n,m,k}\}$. By Ramsey's theorem there is an infinite subset S of ω and an element $g \in F$ such that, for all $n, m, k \in S$, with $n < m < k$, we have $g \in M_{n,m,k}$. For such n, m, k we have either that $l_n \leqslant g \leqslant p_m$ or $p_m \leqslant g \leqslant u_k$. In particular there is an element $n \in S$ for which either $g \leqslant p_n$ or $p_n \leqslant g$. We suppose that $g \leqslant p_n$; the other case is dual. Now, let $S_1 = \{m \in S: n < m\}$. S_1 is an infinite subset of S. If m, k are any two elements of S_1 with $m < k$, we must have $l_m \leqslant g \leqslant p_k$. For, let j be any member of S with $k < j$. Considering the triple m, k, j we have either that $l_m \leqslant g \leqslant p_k$ or $p_k \leqslant g \leqslant u_j$. But the second of these cannot hold: it implies that $p_k \leqslant g \leqslant p_n$, whereas $\{p_i: i \in \omega\}$ is an antichain. Therefore $l_m \leqslant g \leqslant p_k$. We thus have that S_1 is an infinite subset of ω such that, for all m, k in S_1 with $m<k$, $l_m \leq g \leq p_k$. From here we derive a contradiction exactly as in Case 2. □

If P can be embedded in an ordered set having the finite cutset property then P cannot contain a copy of any of the ordered sets mentioned in Lemma 1.3. We will next describe a simple condition for a given ordered set to contain a copy of one of the ordered sets in Fig. 3 (or their duals). This will be the key to establishing our necessary condition for the existence of extensions which have the finite cutset property. We will use the following notation. If $x \in P$ we let

$$x^+ = \{p \in P\colon x \leq p\} \quad \text{and} \quad x^- = \{p \in P\colon p \leq x\}.$$

Lemma 1.4. *Let P be an ordered set and let $x \in P$. Let A be a subset of $I(x)$ such that the family of sets $\{x^+ \cap p^+\colon p \in A\}$ is infinite. Then there is a subset $\{p_n\colon n \in \omega\}$ of A and elements u_n of P, for each $n \in \omega$, such that $\{x\} \cup \{p_n\colon n \in \omega\} \cup \{u_n\colon n \in \omega\}$ is isomorphic to one of the ordered sets in Fig. 3.*

Proof. First we select an infinite subset $\{p_n\colon n \in \omega\}$ of A such that the sets $x^+ \cap p_n^+$ are all distinct. Now $\{p_n\colon n \in \omega\}$ either contains an infinite chain or an infinite antichain. So without loss of generality we may assume that $\{p_n\colon n \in \omega\}$ itself is either an ω-chain, an ω^*-chain or is an antichain. We consider these three cases separately.

Case 1. $\{p_n\colon n \in \omega\}$ is an ω-chain. That is, we suppose that $p_n \leq p_m \leftrightarrow n \leq m$. In this case we have that $n<m \to x^+ \cap p_m^+ \subseteq x^+ \cap p_n^+$. Since the sets $x^+ \cap p_n^+$ are distinct, we can choose for each n, an element $u_n \in x^+ \cap p_n^+ - p_{n+1}^+$. We note that $n<m \to u_n \ngeq u_m$. For if $u_n \geq u_m$ we would have $n_n \geq n_m \geq p_m \geq p_{n+1}$, whereas $u_n \notin p_{n+1}^+$. Now partition the pairs (n, m) of elements of ω with $n<m$ into two boxes B_1 and B_2 as follows:

$(n, m) \in B_1 \leftrightarrow u_n < u_m$;

$(n, m) \in B_2 \leftrightarrow u_n$ is noncomparable to u_m.

Applying Ramsey's theorem we get an infinite subset $S=\{m_0, m_1, m_2, \ldots\}$ of ω, with $m_0<m_1<\cdots$ which is homogeneous for either B_1 or B_2. If S is homogeneous for B_1, then $\{x\} \cup \{p_{m_n}\colon n=0, 1, \ldots\} \cup \{u_{m_n}\colon n=0, 1, \ldots\}$ is isomorphic to Fig. 3(a), while if S is homogeneous for B_2 the set $\{x\} \cup \{p_{m_n}\colon n=0, 1, \ldots\} \cup \{u_{m_n}\colon n=0, 1, \ldots\}$ is isomorphic to Fig. 3(b).

Case 2. $\{p_n\colon n \in \omega\}$ is an ω^*-chain. This is the dual of Case 1 and we find a subset of the required form which is isomorphic either to Fig. 3(c) or to Fig. 3(d).

Case 3. $\{p_n\colon n \in \omega\}$ is an antichain. In this case we first partition the pairs (n, m) with $n, m \in \omega$ and $n<m$ into three boxes B_1, B_2 and B_3 as follows:

$(n, m) \in B_1 \leftrightarrow x^+ \cap p_n^+ \subseteq x^+ \cap p_m^+$;

$(n, m) \in B_2 \leftrightarrow x^+ \cap p_m^+ \subseteq x^+ \cap p_n^+$;

$(n, m) \in B_3 \leftrightarrow x^+ \cap p_n^+$ and $x^+ \cap p_m^+$ are noncomparable under inclusion.

By Ramsey's theorem there is an infinite subset S of ω which is homogeneous for one of B_1, B_2 or B_3. Without loss of generality, and to simplify notation, we may as well assume that $S=\omega$. We now consider each of the three cases.

Case 3(i). Suppose that $n<m\to x^+\cap p_n^+\subseteq x^+\cap p_m^+$. In this case, for each $n=1,2,\ldots,$ choose an element $u_n\in x^+\cap p_n^+-p_{n-1}^+$. Note that $n<m\to u_n\nleqslant u_m$. This is because $u_m\notin p_n^+$, whereas $p_n\leqslant u_n$. Thus we can partition the pairs (n,m) with $1\leqslant n<m$ into two boxes C_1 and C_2 as follows:

$(n,m)\in C_1\leftrightarrow u_n$ is noncomparable to u_m;

$(n,m)\in C_2\leftrightarrow u_m<u_n$.

Applying Ramsey's theorem we get a subset $T=\{m_0,m_1,\ldots.\}$ of ω with $m_0<m_1<\cdots$ which is homogeneous for either C_1 or C_2. If T is homogeneous for C_1 it is clear that $\{x\}\cup\{p_{m_n}: n\in\omega\}\cup\{u_{m_n}: n\in\omega\}$ is isomorphic to Fig. 3(h), while if T is homogeneous for C_2, then $\{x\}\cup\{p_{m_n}: n\in\omega\}\cup\{u_{m_n}: n\in\omega\}$ is isomorphic to Fig. 3(f).

Case 3(ii) Suppose that $n<m\to x^+\cap p_m^+\subseteq x^+\cap p_n^+$. This case is dual to Case 3(i) and we obtain a subset isomorphic either to Fig. 3(e) or Fig. 3(i).

Case 3(iii). Suppose that the family of sets $\{x^+\cap p_n^+: n\in\omega\}$ forms an antichain under inclusion. Recall that we are also assuming that $\{p_n: n\in\omega\}$ is an antichain in P. Now for each pair of integers $n,m\in\omega$ with $n<m$ we can choose an element $u_{n,m}\in x^+\cap p_n^+-p_m^+$. Note that no p_k is $\geqslant$ any of the elements $u_{n,m}$ because p_k is noncomparable to x whereas $u_{n,m}\geqslant x$. Now we can partition the set of all triples (n,m,k) with $n,m,k\in\omega$ and $n<m<k$ into four boxes B_1, B_2, B_3, B_4 as follows. We set

$(n,m,k)\in B_1\leftrightarrow p_n\leqslant u_{m,k}$ and $p_k\leqslant u_{n,m}$;

$(n,m,k)\in B_2\leftrightarrow p_n\leqslant u_{m,k}$ and p_k is noncomparable to $u_{n,m}$;

$(n,m,k)\in B_3\leftrightarrow p_n$ is noncomparable to $u_{m,k}$ and $p_k\leqslant u_{n,m}$;

$(n,m,k)\in B_4\leftrightarrow p_n$ is noncomparable to $u_{n,k}$ and p_k is noncomparable to $u_{n,m}$.

Applying Ramsey's theorem, there is an infinite subset $S=\{m_0,m_1,\ldots.\}$ of ω with $m_0<m_1<\cdots$ which is homogeneous for one of B_1, B_2, B_3 or B_4. We treat each possibility separately as subcases 1, 2, 3, and 4 respectively.

Subcase 1. Suppose S is homogeneous for B_1. In this case, consider the elements $\{u_{m_0,m_1}, u_{m_1,m_2}, u_{m_2,m_3},\ldots\}$. We claim that this set forms an antichain. For, suppose $k<l$. We cannot have $u_{m_k,m_{k+1}}\geqslant u_{m_l,m_{l+1}}$ because $p_{m_{k+1}}\leqslant u_{m_l,m_{l+1}}$ but $p_{m_{k+1}}\nleqslant u_{m_k,m_{k+1}}$. And we cannot have $u_{m_k,m_{k+1}}<u_{m_l,m_{l+1}}$ because $p_{m_{l+1}}\leqslant u_{m_k,m_{k+1}}$ but $p_{m_{l+1}}\nleqslant u_{m_l,m_{l+1}}$. This proves our claim. Now for each $k=1,2,\ldots,$ let $P_k=p_{m_k}$ and let $U_k=u_{m_{k-1},m_k}$. Because S is homogeneous for B_1 (and because of the definition of $u_{i,j}$) we see that $P_k\leqslant U_l\leftrightarrow k\neq l$. Thus the subset $\{x\}\cup\{P_k: k=1,2,\ldots\}\cup\{U_k: k=1,2,\ldots\}$ is isomorphic to Fig. 3(j).

Subcase 2. Suppose S is homogeneous for B_2. In this case, for each $k=0,1,2,\ldots,$ we set $P_k=p_{m_k}$ and $U_k=u_{m_k,m_{k+1}}$. We note that $n\leqslant m\to P_n\leqslant U_m$ and $n>m\to P_n$ is noncomparable to U_m. Also observe that $k<l\to U_k\nleqslant U_l$. This

follows from the fact that $p_{m_l} \leq U_l$ whereas $p_{m_l} \nleq U_k$. Now partition the pairs (k, l) with $k < l$ into two boxes D_1 and D_2 as follows:

$(k, l) \in D_1 \leftrightarrow U_k < U_l$;

$(k, l) \in D_2 \leftrightarrow U_k$ is noncomparable to U_l.

There is an infinite subset $T = \{k_0, k_1, \ldots\}$ of ω, with $k_0 < k_1 < \cdots$, which is homogeneous for either D_1 or D_2. if T is homogeneous for D_1, then the subset $\{x\} \cup \{P_{k_0}, P_{k_1}, \ldots\} \cup \{U_{k_0}, U_{k_1}, \ldots\}$ is isomorphic to Fig. 3(e), while if T is homogeneous for D_2 this subset is isomorphic to Fig. 3(i).

Subcase 3. Suppose S is homogeneous for B_3. In this case, for each $k = 0, 1, 2, \ldots$, we let $P_k = p_{m_{2k}}$ and $U_k = u_{m_{2k}, m_{2k+1}}$. We note that $k \leq l \rightarrow P_l \leq U_k$ and that $k > l \rightarrow P_l$ is noncomparable to U_k. Therefore $k < l \rightarrow U_k \nleq U_l$. Partition the pairs (k, l) with $k < l$ into two boxes E_1 and E_2 as follows:

$(k, l) \in E_1 \leftrightarrow U_k \geq U_l$;

$(k, l) \in E_2 \leftrightarrow U_k$ is noncomparable to U_l.

Arguing as in Subcase 2 we obtain a subset of the required form which is isomorphic either to Fig. 3(f) or to Fig. 3(h).

Subcase 4. Suppose S is homogeneous for B_4. In this case, for each $k = 0, 1, 2, \ldots$, let $P_k = p_{m_k}$ and let $U_k = u_{m_k, m_{k+1}}$. We have that $P_k < U_k$ and P_k is noncomparable to U_l if $k \neq l$. It thus follows that $\{U_k\colon k = 0, 1, 2, \ldots\}$ is an antichain and that the subset $\{x\} \cup \{P_k\colon k \in \omega\} \cup \{U_k\colon k \in \omega\}$ is isomorphic to Fig. 3(g). □

In attempting to compile a minimal list of forbidden configurations which characterize the existence of extensions having the finite cutset property, one might initially hope that the ordered sets described in Lemma 1.3 were sufficient. However, it is easy to see that they are not. For one can show that the ordered set in Fig. 5 has no extensions satisfying the finite cutset property. However this ordered set does not contain any of the configurations described in Lemma 1.3.

In addition to Fig. 5 we can construct other 'bad' configurations by using three-element chains, or four-element chains, etcetera, to 'go around x', just as Fig. 5 uses two-element chains. However even if we take all such configurations, together with those of Lemma 1.3, we will still not have a sufficient number of forbidden configurations to characterize when an ordered set has an extension satisfying the finite cutset property. This is because all such ordered sets are countable, whereas as we will show in Section 2, it is possible that an ordered set P has no extension satisfying the finite cutset property even though every countable subset of P does have such an extension.

Theorem 1.5. *Suppose P can be embedded in an ordered set which has the finite cutset property. Let $x \in P$ and let A be an infinite subset of $I(x)$. Then either there is an infinite subset A_1 of A such that $x^+ \cap p^+ = x^+ \cap q^+$ for all $p, q \in A_1$, or there is an infinite subset A_1 of A such that $x^- \cap p^- = x^- \cap q^-$ for all $p, q \in A_1$.*

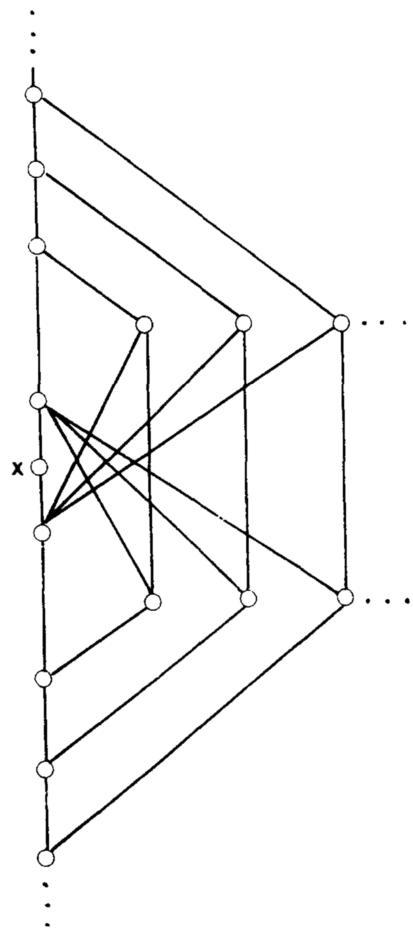

Fig. 5. This ordered set cannot be embedded in one which has the finite cutset property, and does not contain a copy of any of the ordered sets described in Lemma 1.3.

Proof. Assume not. Then in particular the family of sets $\{x^+ \cap p^+ : p \in A\}$ must be infinite, and so by Lemma 1.4 there is an infinite subset $\{p_0, p_1, \ldots\}$ of A and elements $u_0, u_1, \ldots$ in P such that $\{x\} \cup \{p_n : n \in \omega\} \cup \{u_n : n \in \omega\}$ is isomorphic to one of the ordered sets in Fig. 3. Now, the family of sets $\{x^- \cap p_n^- : n \in \omega\}$ must be infinite, otherwise infinitely many of these sets $x^- \cap p_n^-$ would be the same. Applying the dual of Lemma 1.4 there is a subsequence $\{p_{m_0}, p_{m_1}, \ldots\}$ of $\{p_n : n \in \omega\}$, and elements $\{l_{m_0}, l_{m_1}, \ldots\}$ in P such that $\{x\} \cup \{p_{m_n} : n \in \omega\} \cup \{l_{m_n} : n \in \omega\}'$ is isomorphic to the dual of one of the ordered sets in Fig. 3. But now Lemma 1.3 implies that the ordered set $P_1 = \{x\} \cup \{p_{m_n} : n \in \omega\} \cup \{u_{m_n} : n \in \omega\} \cup \{l_{m_n} : n \in \omega\}$ cannot be embedded in an ordered set which has the finite cutset property. But this contradicts our assumption about P. $\square$

We remark that we cannot conclude in Theorem 1.5 that there is an infinite subset A_1 of A for which $x^+ \cap p^+ = x^+ \cap q^+$, for all $p, q \in A_1$, and there is an infinite subset A_2 of A for which $x^- \cap p^- = x^- \cap q^-$, for all $p, q \in A_2$. This can be easily seen by referring to Fig. 3(f) above. The ordered set in Fig. 6 has the finite cutset property and contains a copy of Fig. 3(f).

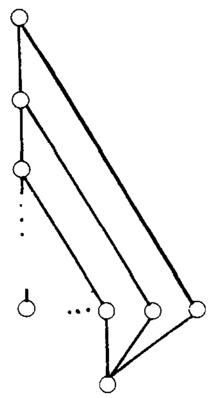

Fig. 6.

We will now show that the existence of an extension satisfying the finite cutset property does have consequences of both an up and a down nature for *uncountable* sets.

Theorem 1.6. *Suppose P can be embedded in an ordered set with the finite cutset property. Let A be an uncountable antichain in P. Then there exists an element $x \in A$ and an infinite subset A_1 of A such that $x^+ \cap p^+ = x^+ \cap q^+$, for all $p, q \in A_1$.*

Proof. Suppose Q has the finite cutset property and that $P \subseteq Q$. Then for every $x \in A$ there is a finite set $F(x) \subseteq Q$ such that all elements of $F(x)$ are noncomparable to x and such that every maximal chain in Q meets $\{x\} \cup F(x)$. For each $n \in \omega$, let $A_n = \{x \in A\colon |F(x)| = n\}$. Since $A = \bigcup_{n \in \omega} A_n$, one of the sets A_n must be uncountable. If A_{n_0} is such a set, we may replace the original set A by a subset of A_{n_0} of cardinality ω_1. Thus, we may suppose without loss of generality that $|A| = \omega_1$, and that, for some fixed $n_0 \in \omega$, $|F(x)| = n_0$ for all $x \in A$.

Now, let x be any fixed element of A, and let $B_x = \{p \in A$: there is no element $f \in F(x)$ such that $f \leqslant p\}$. We claim that the family of sets $\{x^+ \cap p^+\colon p \in B_x\}$ is finite. For otherwise, by Lemma 1.4 we can find a subset $\{p_n\colon n \in \omega\}$ of B_x and elements $\{u_n : n \in \omega\}$ of P such that $\{x\} \cup \{p_n\colon n \in \omega\} \cup \{u_n\colon n \in \omega\}$ is isomorphic to one of the ordered sets in Fig. 3. Now, by definition of B_x, any member of $F(x)$ which belongs to a maximal chain of Q containing any p_n must be above that p_n. However, as a second glance at the proof of Lemma 1.3 shows, it is not possible to meet all the maximal chains in Q containing various pairs $\{p_n, u_m\}$ with a finite number of elements which all lie between the p's and u's. Therefore, indeed $\{x^+ \cap p^+\colon p \in B_x\}$ is finite. Now, if B_x is infinite, there would be an infinite subset B_1 of B_x for which $x^+ \cap p^+ = x^+ \cap q^+$, for all $p, q \in B_1$. This would give the conclusion desired in the theorem. So we may as well suppose that B_x is finite for every $x \in A$.

Since A is uncountable we can inductively choose elements $a_\alpha \in A$ for $\alpha < \omega_1$ such that $a_\alpha \notin \bigcup_{\beta < \alpha} B_{a_\beta}$, for all $a < \omega_1$. The result is a subset $\{a_\alpha\colon \alpha < \omega_1\}$ of A with the property that, for all $\alpha, \beta < \omega_1$ with $\alpha < \beta$, there is an element $f \in F(a_\alpha)$ such that $f \leqslant a_\beta$. Now, for each $\alpha < \omega_1$, list the elements of the set $F(a_\alpha)$ as

$F(a_\alpha) = \{f_{\alpha,1}, f_{\alpha,2}, \ldots, f_{\alpha,n_0}\}$. We now partition the set of all pairs (α, β) with $\alpha, \beta < \omega_1$ and $\alpha < \beta$ into boxes $C_1, C_2, \ldots, C_{n_0}$ as follows: for $i = 1, 2, \ldots, n_0$ we set $(\alpha, \beta) \in C_i \leftrightarrow f_{\alpha,i} \leq a_\beta$. Employing the partition relation $\omega_1 \rightarrow (\omega + 1)^2_{n_0}$ (see [2]), we can infer the existence of a subset $\{\alpha_0, \alpha_1, \ldots, \alpha_\omega\}$ of ω_1 with $\alpha_0 < \alpha_1 < \cdots < \alpha_\omega$, and an integer $i \in \{1, 2, \ldots, n_0\}$ such that $n < m \leq \omega \rightarrow f_{\alpha_n,i} \leq a_{\alpha_m}$. We note that if $n < m < \omega$, then $a^-_{\alpha_n} \cap a^-_{\alpha_\omega} \neq a_{\alpha_m} \cap a^-_{\alpha_\omega}$. Indeed, $f_{\alpha_n,i} \in a^-_{\alpha_m} \cap a^-_{\alpha_\omega} - a^-_{\alpha_n}$. We now apply Theorem 1.5 to $x = a_{\alpha_\omega}$ and the set $\{a_{\alpha_n}: n \in \omega\} \subseteq I(x)$, and conclude that there is an infinite subset S of ω such that $a^+_{\alpha_n} \cap a^+_{\alpha_\omega} = a^+_{\alpha_m} \cap a^+_{\alpha_\omega}$, for all $n, m \in S$. This is the desired conclusion. $\square$

By duality, there are really two conclusions we can draw from Theorm 1.6, and so we obtain the following corollary.

Corollary 1.7. *Suppose P can be embedded in an ordered set which has the finite cutset property. Let A be an uncountable antichain in P. Then there exists an element $x \in A$ and infinite subsets A_1, A_2 of A such that $x^+ \cap p^+ = x^+ \cap q^+$, for all $p, q \in A_1$, and $x^- \cap p^- = x^- \cap q^-$, for all $p, q \in A_2$.*

Proof. Let $A_0 = \{x \in A$: there is an infinite subset S of A such that $x^+ \cap p^+ = x^+ \cap q^+$, for all $p, q \in S.\}$ We note that $A - A_0$ is countable; otherwise we could apply Theorem 1.6 to the uncountable antichain $A - A_0$. In particular, A_0 is uncountable. By duality (applying Theorem 1.6 to the antichain A_0) there is an element $x \in A_0$ for which there is an infinite set $T \subseteq A_0$ such that $x^- \cap p^- = x^- \cap q^-$, for all $p, q \in T$. This x is the desired element. $\square$

We have three remarks concerning these results. Firstly, we cannot expect in general to obtain from Theorem 1.6 an uncountable subset A_1 of A and an element $x \in A$ with $x^+ \cap p^+ = x^+ \cap q^+$, for all $p, q \in A_1$. Nor can we expect in general to obtain sets A_1, A_2 from Corollary 1.7 with $A_1 = A_2$. This is shown by the ordered set P in Fig. 1 (which already has the finite cutset property.) This latter example also shows that there may not exist four distinct elements x, p, q, r in A with $x^+ \cap p^+ = q^+ \cap r^+$.

Finally, we note that the conclusions of Theorem 1.6 and Corollary 1.7 hold for any uncountable subset A of P, whether or not A is an antichain. This follows from the fact that A will either contain an uncountable antichain or will contain an element x for which the sets $\{p \in A: p < x\}$ and $\{p \in A: x < p\}$ are infinite.

2. Countable subsets do not determine the existence of an extension having the finite cutset property

We will now make use of Theorem 1.6 to describe the example promised above.

Example 2.1. Let X be an uncountable set and let P be the set of all one and two-element subsets of X, ordered by inclusion. That is, $P = \{x \in P(X)\colon 1 \leqslant |x| \leqslant 2\}$. Then P cannot be embedded in an ordered set having the finite cutset property, whereas every countable subset of P can.

Proof. The fact that P cannot be embedded in an ordered set with the finite cutset property follows immediately from Theorem 1.6, since P contains an uncountable antichain A (the one-element subsets of X) such that $a^+ \cap b^+ \neq a^+ \cap c^+$, for all distinct $a, b, c \in A$. The fact that every countable subset of P does have such an extension is a special case of the following lemma.

Lemma 2.2. *Let P be a countable ordered set of length 2. Then P can be embedded in an ordered set which has the finite cutset property.*

Proof. Recall that by the length of P we mean the supremum of cardinalities of the chains in P. Our assumption means that evey element of P is either a maximal element of P or a minimal element of P. Let $P_1 = \{x \in P\colon x$ is maximal and minimal in $P\}$ and let $P_2 = P - P_1$. Note that no element of P_1 is comparable with any element of P_2.

If we exhibit ordered sets Q_1 and Q_2, both having the finite cutset property, such that $P_1 \subseteq Q_1$ and $P_2 \subseteq Q_2$, we can then form the unordered sum $Q = Q_1 + Q_2$ to obtain the desired extension of P. But P_1 is an antichain, so it is easy to find an extension Q_1 for P_1: if P_1 is finite, take $Q_1 = P_1$, while if P_1 is infinite use the ordered set in Fig. 1 above. Thus we need only construct an extension for P_2. We can write $P_2 = \{x_n\colon n \in \omega\} \cup \{y_n\colon n \in \omega\}$, where each x_n is a minimal element and each y_n is a maximal element, and where $n \neq m \rightarrow x_n \neq x_m$ and $y_n \neq y_m$. (If the set of maximal elements (or minimal elements) is finite then obvious modifications to our argument apply.) Let $\{l_n\colon n \in \omega\}$ and $\{u_n\colon n \in \omega\}$ be sets of distinct new elements and let $Q = P_2 \cup \{l_n\colon n \in \omega\} \cup \{u_n\colon n \in \omega\}$ be the ordered set depicted in Fig. 7.

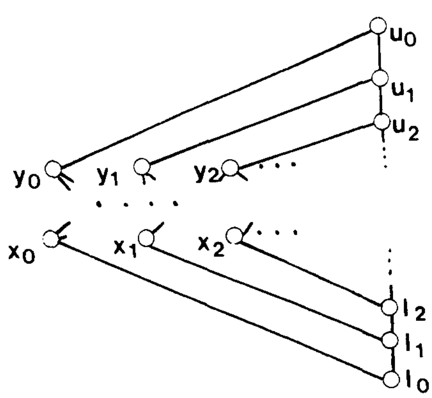

Fig. 7.

Specifically, we order Q in such a way that:

(i) $l_0 < l_1 < l_2 < \cdots < u_2 < u_1 < u_0$;

(ii) $y_n < u_m \leftrightarrow m \leqslant n$;

(iii) $l_m < x_n \leftrightarrow m \leqslant n$;

(iv) $x_n < u_m$ for all $n, m \in \omega$;

(v) $l_m < y_n$ for all $n, m \in \omega$.

Of course we preserve the original ordering of P_2. Then Q has the finite cutset property. This follows from the fact that the two sets

$$\{x_n\} \cup \{x_i\colon i<n\} \cup \{l_{n+1}\} \quad \text{and} \quad \{y_n\} \cup \{y_i\colon i<n\} \cup \{u_{n+1}\}$$

are cutsets for Q for each $n \in \omega$. We verify this fact for the first of these two sets. So, let $n \in \omega$, and let C be a maximal chain in Q. We wish to show that C intersects the set $F_n = \{x_n\} \cup \{x_i\colon i<n\} \cup \{l_{n+1}\}$. If $x_n \in C$ we are done. If not, either every l_k, for $k \in \omega$, is in C or not. If so, again we are done. If not, let k_0 be the smallest k for which $l_k \notin C$. Note that $k_0 > 0$, since l_0 is the smallest element of Q, hence $l_0 \in C$. If $n+1 < k_0$ then we would have $l_{n+1} \in C$ and again we are done. So we may as well suppose that $k_0 \leqslant n+1$. Now consider l_{k_0-1}. We have $l_{k_0-1} \in C$. Now the set $D = \{p \in C\colon l_{k_0-1} < p\}$ is a maximal chain in the ordered set $R = \{p \in Q\colon l_{k_0-1} < p\}$. And the set $S = \{l_{k_0}, x_{k_0-1}\}$ is a coinitial subset of R. Therefore $D \cap S \neq \emptyset$. Since $l_{k_0} \notin D$ we have $x_{k_0-1} \in D$. Since $k_0 - 1 \leqslant n$ this shows that C meets F_n, as desired. □

We note that the conclusion of Lemma 2.2 does not hold for ordered sets of length 3, as we can see from Lemma 1.3 above. We also point out that another example having the properties in Example 2.1 can be described as follows. Let P be the Cantor tree with a top level attached. That is, let $P = \bigcup_{n<\omega} 2^n \cup 2^\omega$, ordered by inclusion. Note that here 2^n denotes the set of all functions from the set $n = \{k\colon k<n\}$ to the set $2 = \{0, 1\}$. P has no extension with the finite cutset property because the top level 2^ω of P is an antichain which violates the dual of Theorem 1.6 above. However every countable subset of P has such an extension, as suggested in Fig. 8 below.

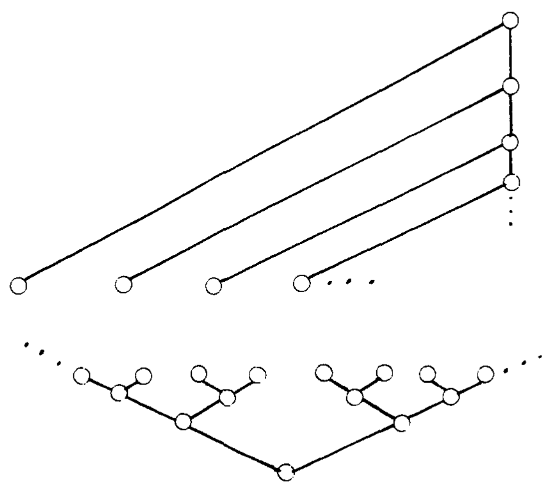

Fig. 8.

The examples in this section suggest a more restricted question than that asked at the beginning of this paper, one which we have been unable to answer, namely: characterize (by means of a minimal list of forbidden configurations), those countable ordered sets which have an extension satisfying the finite cutset property.

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